MATH 4060 MIDTERM EXAM (FALL 2018)

Name: ____ Student ID: ___________________________

Answer all questions. Write your answers on this question paper.
No books, notes or calculators are allowed. Time allowed: 105 minutes.

Q1(a)	Q1(b)	Q2(a)	Q2(b)(i)	Q2(b)(ii)	Q3	Q4(a)	Q4(b)	Q5	Total
/3	/9	/12	/12	/12	/12	/12	/12	/16	/100

1. Suppose f is holomorphic on an open set containing the closed strip $S := \{z \in \mathbb{C} \colon |\operatorname{Im} z| \leq 1\}$, and suppose that there exists a constant $A \in \mathbb{R}$ such that

$$|f(z)| \leq \frac{A}{1+|z|^2} \quad \text{for every } z \in S.$$

   (a) Let $-1 \leq a \leq b \leq 1$. Show that

$$\lim_{R \to +\infty} \left(\int_R^\infty |f(x+ia)| dx + \int_a^b |f(R+iy)| dy + \int_R^\infty |f(x+ib)| dx \right) = 0.$$

   (3 points)

(b) Let

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

be the Fourier transform of f. Show that there exists a constant $B \in \mathbb{R}$ such that

$$|\widehat{f}(\xi)| \leq B e^{-2\pi|\xi|} \quad \text{for every } \xi \in \mathbb{R}.$$

You should indicate where you used the holomorphicity assumption on f. (9 points)

2. Let f be a holomorphic function on the punctured open disc $\mathbb{D}\setminus\{0\}$ where $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$.

   (a) Show that f admits a convergent Laurent series expansion on $\mathbb{D} \setminus \{0\}$, i.e. there exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers such that

   $$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

   for every $z \in \mathbb{D} \setminus \{0\}$. (You may use Cauchy's theorem freely.) (12 points)

(b) Prove the following statements about the singularity of f at $z = 0$:

(i) If there exist real numbers $\varepsilon > 0$ and $A > 0$ such that

$$|f(z)| \leq \frac{A}{|z|^{1-\varepsilon}} \quad \text{for all } z \in \mathbb{D} \setminus \{0\},$$

then 0 is a removable singularity of f, i.e. there exists a holomorphic function g on $\mathbb{D}$ such that $g = f$ on $\mathbb{D} \setminus \{0\}$. (12 points)

(ii) If 0 is an essential singularity of f, i.e. if there are infinitely many negative integers n for which the coefficients a_n in part (a) is non-zero, then the image of $\mathbb{D} \setminus \{0\}$ under f is dense in $\mathbb{C}$. (12 points)

You may use freely Cauchy's theorem / Morera's theorem / your proof of part (a) of the question. You may also use the statement (i) in the proof of statement (ii), and vice versa, but your argument should NOT be circular; in particular, some points will be deducted if you only prove that (i) is equivalent to (ii), without actually proving any of them. Please use also the next page if you need more space.

3. Let $\mathbb{D}$ be the open unit disc $\{z \in \mathbb{C} \colon |z| < 1\}$, and let f be a non-constant holomorphic function in an open set containing the closure of $\mathbb{D}$. Suppose $|f(z)| = 1$ whenever $|z| = 1$. Show that every point in $\mathbb{D}$ lies in the image of $\mathbb{D}$ under f, i.e. $\mathbb{D} \subseteq f(\mathbb{D})$. (12 points)

4. For each of the following statements, determine whether it is true or false. Justify your assertion.

   (a) There exists an entire function f, not identically zero and of order of growth $\leq 1/2$, such that $f(n^2) = 0$ for every integer n. (12 points)

(b) Let f be a complex-valued function defined on an open set $\Omega \subseteq \mathbb{C}$, and let $g(z) = [f(z)]^2$ and $h(z) = [f(z)]^3$ for every $z \in \Omega$. If g and h are both holomorphic on Ω, then f is holomorphic on Ω. (12 points)

5. Let f be a holomorphic function on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. Suppose f is bounded on $\mathbb{D}$, i.e. there exists a constant $A \in \mathbb{R}$ such that $|f(z)| \leq A$ for every $z \in \mathbb{D}$, and suppose f vanishes at $\frac{n}{n+1}$ for every $n \in \mathbb{N}$. Show that f must be identically zero on $\mathbb{D}$. (16 points)